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LETTER TO THE EDITOR

Vacant animals and 2D to 1D crossover in the percolation problem

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Abstract. We present a discussion of the appearance of one-dimensional (1D) effects in narrow strips of percolating systems. Using both scaling and probabilistic arguments, rather than specific models of the infinite percolating cluster, we predict that 1D effects are observable at strip lengths $L_x > L_y \exp(L_y/\xi_2)^{D_A}$ where D_A is the fractal dimensionality of very large clusters below the percolation threshold.

The 2D to 1D crossover of a percolating system was recently discussed (Deutscher 1982) using the nodes and links model (Skal and Shklovskii 1974, de Gennes 1976) for the infinite cluster, and known 1D results. In this model, 1D effects were predicted to occur at strip widths $\xi_2 \ln(L_x/\xi_2)$.

We present here a more general discussion of the same problem, based uniquely on scaling and probabilistic arguments rather than on any specific model of the infinite cluster. We also extend the discussion to the situation below the percolation threshold.

The crossover is predicted to occur when $L_y \leqslant \xi_2[\ln(L_x/L_y)]^{1/D_A}$, which is a more restrictive condition than that obtained with the nodes and links model.

For a one-dimensional (1D) percolation problem the cluster numbers are given by (Wortis 1975)

$$n_s(p) = p^s(1-p)^2 \tag{1}$$

Exponentiating this expression and assuming a finite value of conductance for each unremoved bond, we may write an expression for the conductivity on length scale s (all lengths here are measured in units of the microscopic length d),

$$\sigma(s) \sim \exp(-s/\lambda) \tag{2}$$

with $1/\lambda = -\ln p$ (Bernasconi and Schneider 1981).

Let us now consider a 2D strip of finite width L_y. On large length scales this strip may be viewed as essentially 1D. We are aiming at evaluating the characteristic length $\tilde{\lambda}$ (measured along the strip) that replaces λ in equation (2). On length scales larger than $\tilde{\lambda}$ the exponential decrease of the conductivity becomes apparent. We divide our strip into squares of linear size L_y (figure 1). In other terms, we consider that the strip is composed of $L_y \times L_y$ squares that have been cut out independently from the infinite 2D plane, and put side by side. The conductivity of a segment of length L_x

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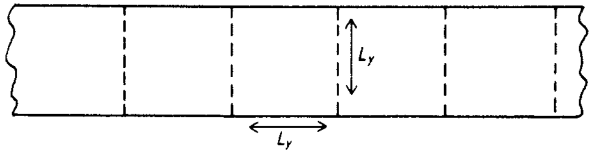

Figure 1. A finite width strip divided into squares of $L_y \times L_y$.

of the strip can be written as

$$\sigma(L_x) = \sigma(L_y)(\alpha \cdot C)^{L_x/L_y}. \tag{3}$$

Here $\sigma(L_y)$ is the average conductivity of an $L_y \times L_y$ square. C is the contact probability of such a square, i.e. the probability of having (at least) one cluster connecting both sides of the square. α is the conditional probability for two neighbouring squares, each having a contact between both of its edges, to be connected. In writing equation (3) we assumed that the corresponding factors for each square are independent. Moreover, it is assumed that if there is contact in the system the conductance may be calculated using Ohm's law, e.g. by simple juxtaposition of $L_y \times L_y$ squares.

Let us now try to evaluate the various factors appearing in equation (3). Hereafter all lengths are measured in units of the lattice spacing d. We denote by ξ_2 the percolation correlation length of a 'real' 2D system, $\xi_2 \sim (p-p_c)^{-\nu}$. All the quantities here and in the following, e.g. p_c and ν, refer to 2D systems. The value of ν is 1.33. (For recent reviews on percolation, see Stauffer (1979), Essam (1980).) In principle, we have four asymptotic regimes, defined by p being above (below) p_c, and ξ_2 being greater (smaller) than L_y.

For $\sigma(L_y)$ we may use the following scaling expressions (Y Kantor, unpublished), valid for p close to p_c and $L_y \gg 1$,

$$\sigma_{p>p_c}(L_y) \sim L_y^{-t/\nu} f(L_y^{1/(p-p_c)}). \tag{4}$$

Here, t is the conductivity exponent defined by $\sigma \sim (p-p_c)^t$ for a 2D problem. The asymptotic behaviour of the scaling functions is $f(x \to 0) \to \text{constant}$, $f(x \to +\infty) \to x^t$, $f(x \to -\infty) \to 0$.

For $p < p_c$ and $\xi_2 \ll L_y$ the contact (if any) within a square is due to a very rare 'animal' cluster, whose fractal dimensionality D_A (Mandelbrot, 1977, 1982) relates its radius to its mass M by $M \sim (\text{radius})^{D_A}$. The best numerical estimate for D_A is 1.5605 ± 0.0007 (Derrida and de Sèze (1982); see also Stauffer (1978), Herrmann (1979). Notice that this is quite different from the value predicted by Parisi and Sourlas (1981), $D_A = 1/0.61 = 1.64$.) The cluster numbers of those rare 'animals' are $\sim\exp(-\text{constant } M)$. We consider clusters whose linear size exceeds L_y. Expressing M in terms of the cluster's radius, and assuming that for $\xi_2 \gg 1$ the contact is function of a scaling argument L_y/ξ_2 (see Reynolds *et al* 1980), we arrive at the expression

$$C_{p<p_c}(\xi_2 \ll L_y) \sim \exp[-(L_y/\xi_2)^{D_A}]. \tag{5a}$$

Hereinafter we omit constants in the exponents and ignore slowly varying coefficients.

Since bond percolation on a square lattice is a self-dual problem, we may write an expression for the regime $p > p_c$, $L_y > \xi_2$,

$$C_{p>p_c}(\xi_2 \ll L_y) = 1 - \exp[-(L_y/\xi_2)^{D_A}]. \tag{5b}$$

The interpretation of this expression is straightforward. When $p > p_c$ and $L_y > \xi_2$, most $L_y \times L_y$ squares are conducting. However, because of the existence of (rare) clusters of vacant bonds ('vacant animals') of linear size $\sim L_y$, there is a small but finite probability that a given square will be interrupted. These rare clusters play a crucial role in the 2D to 1D crossover. In particular, they justify our description of the strip as composed of squares of size L_y rather than of bands of length ξ_2 (Deutscher 1982): most interruptions of the strip are due to clusters of vacant bonds of a size only slightly larger than L_y, because larger clusters are even more rare. Hence, they occupy a length of the order of L_y. In other words, in order to find another independent vacant animal along the strip, we must go at least a distance of the order of L_y (and not ξ_2).

The validity of equations (5) is presumably more general than for a square lattice only.

There are indications that for L_y not too large (compared with ξ_2) C is analytic in $p - p_c$, i.e. analytic in $\xi_2^{1/\nu}$ (Reynolds *et al* 1980). This means that for $p \simeq p_c$, $L_y < \xi_2$, C could be expanded in powers of $(L_y/\xi_2)^{1/\nu}$,

$$C_{p>p_c}(\xi_2 \gg L_y) = \tfrac{1}{2} + (L_y/\xi_2)^{1/\nu}, \qquad (5c)$$

$$C_{p<p_c}(\xi_2 \gg L_y) = \tfrac{1}{2} - (L_y/\xi_2)^{1/\nu}. \qquad (5d)$$

For $L_y > \xi_2$ the contact function (5c), (5d) may be written by extrapolation as $1 - \exp[-(L_y/\xi_2)^{2/\nu}]$ and $\exp[-(L_y/\xi_2)^{2/\nu}]$ for $p > p_c$, $p < p_c$, respectively. These expressions are an alternative to equations (5a, b), and give (nearly) the same result on the basis of totally different extrapolation. Though it is not obvious that these expressions are valid for $L_y \gg \xi_2$, this may be the case; one notes that the numerical value of $2/\nu$ in 2D is 1.5 which is close to the numerical value for D_A stated above. (Note also the interesting observation that within mean field theories $2/\nu = D_A = 4$ (the value $D_A = 4$ was given by Zimm and Stockmayer (1949).)

A direct estimation of the joining probability α is very complicated, since one cannot neglect the correlations between two neighbouring $L_y \times L_y$ squares. However, it can be shown that α is always larger than C. Thus, the crossover from the 2D to the 1D regime is dominated by the exponential factor C^{L_x/L_y}. Using equation 5(b) we find that the crossover to 1D is expected to occur at length $\tilde{\lambda}$, satisfying

$$\ln(\tilde{\lambda}/L_y) = (L_y/\xi_2)^{D_A} \qquad (p > p_c, L_y \gg \xi_2). \qquad (6)$$

In a typical experiment (Deutscher *et al* 1979) $d \sim 300$ Å and $p - p_c \simeq 0.1$. This leads to $\xi_2 \simeq 0.65$ μm $\simeq 21d$. With a strip of width $L_y \simeq 1.3$ μm $\simeq 42d$ one has $\tilde{\lambda} \simeq 22$ μm, a fairly convenient value. However, since equation (6) is valid only up to a multiplicative constant which is unknown and may not be of order unity (it is related to the relative weight of the animals compared with that of the regular clusters), an experimental determination of the value of $\tilde{\lambda}$ would by no means be a trivial verification of currently accepted scaling theories.

For the other regimes ($p < p_c$ and $p > p_c$, $\xi_2 > L_y$) it follows from our analysis that 1D exponential decay is expected already on length scales $\sim L_y$, hence $\tilde{\lambda} \sim L_y$.

Notice that our analysis, based on scaling arguments, is insensitive to the explicit picture of the infinite cluster. In particular, no use of either the links and nodes picture (Skal and Shklovskii 1974, de Gennes 1976) or self-similar models (Gefen 1981, Gefen *et al* 1981) is necessary. Our result (equation (8)) predicts how the strip length, at which 1D effects appear, scales with the strip width and the percolation correlation length. The main interest of experimental measurements of $\tilde{\lambda}$ would be to determine the weight of the large 'animals', for which no results are available at the moment.

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